The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2

Date: November 10, 2005

Course: EE 313 Evans

Name: Set, Solution
Last, First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework assignments and the homework solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Fully justify your answers unless instructed otherwise.**

Problem	Point Value	Your score	Topic
1	20		Differential Equation
2	20		Discrete-Time Stability
3	20		Discrete-Time Tapped Delay Line
4	24		Analog Filter Design
5	16		Potpourri
Total	100		

L-30

Problem 2.1 Differential Equation. 20 points.

For a continuous-time linear time-invariant (LTI) system with input $x(t)$ and output $y(t)$ governed by the differential equation

$$\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=x(t)$$

for $t \geq 0^-$. LTI implies $y(0^-)=0$ and $y'(0^-)=0$.

(a) What is the transfer function? 6 points.

Take the Laplace transform of both sides:

$$s^2 Y(s) + 4s\,Y(s) + 3\,Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+4s+3} \quad \text{for } Re\{s\} > -1$$

(b) What are the values of the poles and zeroes of the transfer function. 6 points.

Zeros are the roots of the numerator polynomial: no zeros.

Poles are the roots of the denominator polynomial:

$$s^2+4s+3 = (s+1)(s+3) = 0$$

Poles are located at $s=-1$ and $s=-3$.

(Region of convergence is $Re\{s\} > -1$.)

(c) Give a formula for the step response of the system. 8 points.

The step response is the response (output) for an input that is a unit step function.

$$x(t) = u(t) \Longleftrightarrow X(s) = \frac{1}{s} \quad \text{for } Re\{s\} > 0.$$

$$Y(s) = H(s)\,X(s) = \frac{1}{s(s+1)(s+3)} = \frac{a_0}{s} + \frac{b_0}{s+1} + \frac{c_0}{s+3}$$

$$a_0 = \left[s\,H(s)\right]\Big|_{s=0} = \frac{1}{3}$$

$$b_0 = \left[(s+1)\,H(s)\right]\Big|_{s=-1} = -\frac{1}{2}$$

$$c_0 = \left[(s+3)\,H(s)\right]\Big|_{s=-3} = \frac{1}{6}$$

$$y(t) = \left(\frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}\right)u(t)$$

Problem 2.2 Discrete-Time Stability. 20 points.

In this problem, a discrete-time system has an input signal denoted by $x[n]$ and an output signal denoted by $y[n]$.

(a) Is the system defined by $y[n+2]-4y[n]=x[n+2]$ asymptotically stable, marginally stable, or unstable? Why? 8 points.

characteristic roots (poles) impact stability.

characteristic roots are

$$\gamma^2 - 4 = 0 \Rightarrow (\gamma+2)(\gamma-2)=0 \Rightarrow \gamma=-2 \text{ and } \gamma=2.$$

Roots are outside the unit circle $\Rightarrow$ system is unstable.

(b) Let K be a real-value constant. For what values of K is the following system asymptotically stable? $y[n+2]-Ky[n]=x[n+2]$. Why? 8 points.

Characteristic roots are

$$\gamma^2 - K = 0 \Rightarrow \gamma^2 = K \Rightarrow \gamma = \pm\sqrt{K}$$

For the system to be asymptotically stable, all characteristic roots must be inside the unit circle:

$|K| < 1$ or equivalently $-1 < K < 1$.

(c) Rewrite $y[n+2]-4y[n]=x[n+2]$ in causal form. Assuming that the system is linear and time-invariant, compute the first three samples of the impulse response. 4 points

Shift (delay) by two samples via the variable substitution $n = n-2$: $y[n] - 4y[n-2] = x[n]$

The impulse response is the response of the system to an input that is an impulse: $x[n] = \delta[n]$.

Because the system is LTI, $y[-1]=0$ and $y[-2]=0$. That is, initial conditions are zero.

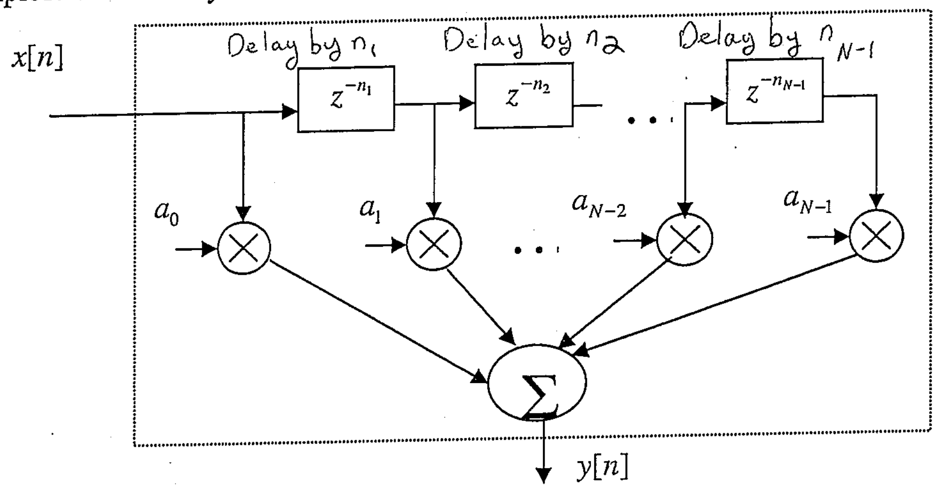
$y[0] = x[0] + 4y[-2] = 1$

$y[1] = x[1] + 4y[-1] = 0$

$y[2] = x[2] + 4y[0] = 4$

Problem 2.3 Discrete-Time Tapped Delay Line. 20 points.

A linear time-invariant (LTI) discrete-time tapped delay line with input $x[n]$, output $y[n]$, and N-1 delay elements is shown below as a block diagram. The notation z^{-m} means a delay of m samples. Each delay element has a different amount of delay.



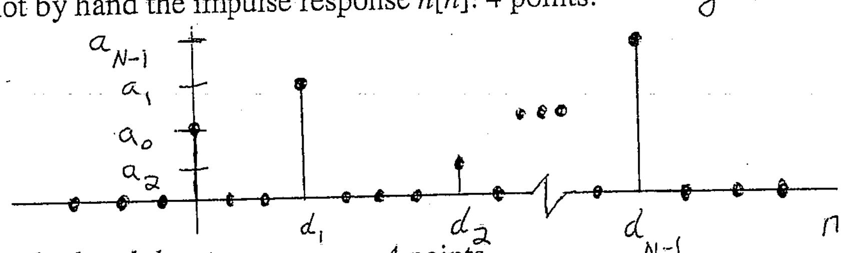
(a) Give a formula for the input-output relationship. 8 points.

Delays add along the delay line. Let $d_m = \sum_{i=0}^{m} n_i$ and $n_0 = 0$.

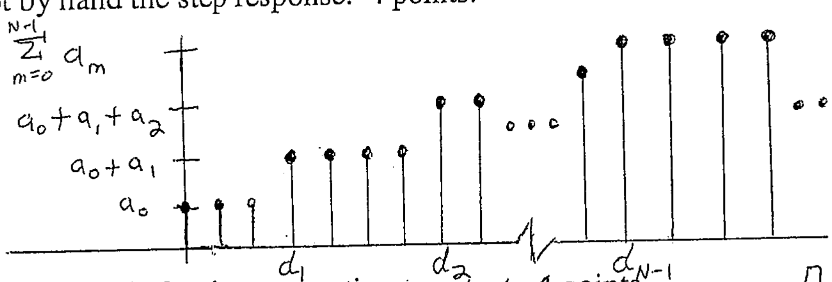
$y[n] = a_0\, x[n] + a_1\, x[n-d_1] + a_2\, x[n-d_2] + \cdots + a_{N-1}\, x[n-d_{N-1}]$

$y[n] = \sum_{m=0}^{N-1} a_m\, x[n-d_m]$

(b) Plot by hand the impulse response $h[n]$. 4 points.



(c) Plot by hand the step response. 4 points.



The steady state value of the step response is $\sum_{m=0}^{N-1} a_m$.

(d) Give a formula for the system time constant. 4 points.

The system time constant is the amount of time to fully respond to an input. The system time constant can be computed as the extent of the impulse response (in samples) minus one, which is equal to $d_{N-1} = n_1 + n_2 + \cdots + n_{N-1}$.

Problem 2.4 Analog Filter Design. 24 points.

A second-order filter has two poles, and 0, 1, or 2 zeros. The transfer function for a second-order analog filter with poles p_0 and p_1 and zeros z_0 and z_1 follows:

$$H(s) = b_0 \frac{(s-z_0)(s-z_1)}{(s-p_0)(s-p_1)} = b_0 \frac{s^2 - (z_0+z_1)s + z_0 z_1}{s^2 - (p_0+p_1)s + p_0 p_1}$$

(a) Assuming that the poles are conjugate symmetric, show that the denominator polynomial has real-valued coefficients. 6 points.

The coefficients of the denominator polynomial are $1, p_0+p_1, p_0 p_1$.

Let $p_0 = a + jb$. The conjugate symmetric value is $p_1 = a - jb$.

$p_0 + p_1 = (a+jb) + (a-jb) = 2a$, which is real-valued.

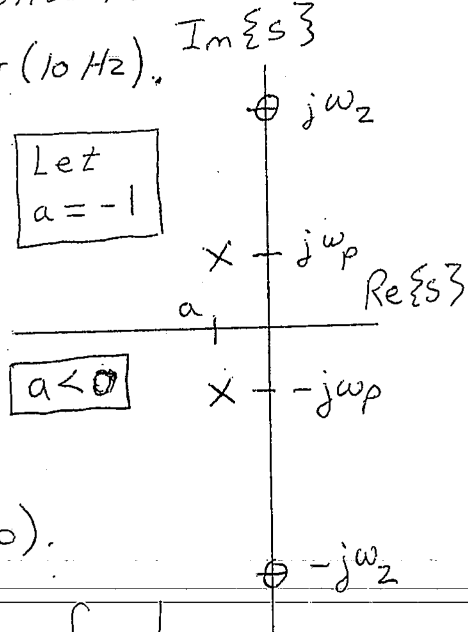
$p_0 p_1 = (a+jb)(a-jb) = a^2 + b^2$, which is real-valued.

(b) Design the transfer function for a second-order analog filter by determining the locations of the two poles and two zeros and determining the scaling constant b_0. The filter is to pass frequencies between –10 Hz and 10 Hz and zero out (notch out) frequencies at -60 Hz and 60 Hz. Your poles should be conjugate symmetric. Your zeros should also be conjugate symmetric. 18 points.

Zeros on the imaginary axis correspond to frequencies that are zeroed out (notched out). $\omega_z = 2\pi(60\text{ Hz})$.

Poles near the imaginary axis correspond to frequencies being passed. $\omega_p = 2\pi(10\text{ Hz})$.

(Lathi's book, pp. 450-451, discusses lowpass filter design in terms of a cutoff frequency, ω_c, which is between ω_p and ω_z.)



$z_0 = j\,2\pi(60)$. $z_1 = -j\,2\pi(60)$.

$p_0 = a + j\,2\pi(10)$. $p_1 = a - j\,2\pi(10)$.

Let $H(j0) = 1 \Rightarrow b_0 \dfrac{z_0 z_1}{p_0 p_1} = 1$. Solve for b_0.

Problem 2.5 Potpourri. 16 points.

(a) *Either prove the following statement to be true, or give a counterexample to show that the following statement is false*: The discrete-time convolution of two finite duration signals always produces a finite duration result that is longer than either of the signals being convolved. 4 points.

(This question is the discrete-time version of problem 5(a) on midterm #1.)

False. $\delta[n] * \delta[n] = \delta[n]$. The result is of length one sample, which is not longer than either of the signals being convolved.

(b) Find the transfer function for the following continuous-time linear time invariant system with input $x(t)$ and output $y(t)$ governed by the differential equation (for $t \geq 0^-$). 4 points.

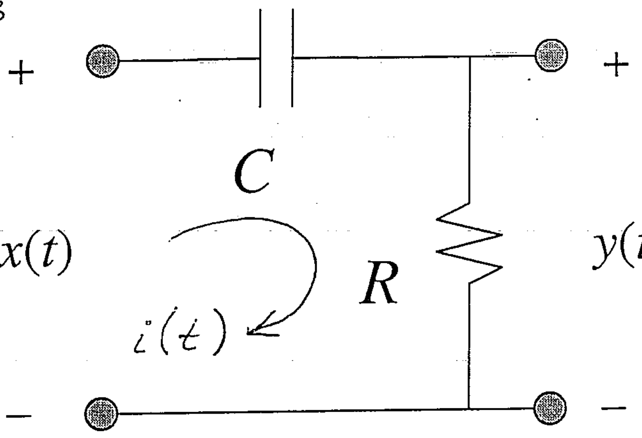
$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + x(t)$$

Be sure to simplify the transfer function as much as possible.

$$s^2 Y(s) + 4s\,Y(s) + 3\,Y(s) = s\,X(s) + X(s)$$

$$H(s) = \frac{Y(z)}{X(z)} = \frac{\cancel{s+1}}{(\cancel{s+1})(s+3)} = \frac{1}{s+3}$$

(c) For the continuous-time analog linear time-invariant circuit on the right with input $x(t)$ and output $y(t)$,



i. Give the transfer function. 4 points.

Impedance $Z(s) = R + \frac{1}{Cs}$

$$I(s) = \frac{X(s)}{Z(s)}$$

$$Y(s) = R\,I(s) = \frac{R}{R + \frac{1}{Cs}} X(s)$$

$$\rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s + \frac{1}{RC}}$$

$Re\{s\} > -\frac{1}{RC}$

ii. What kind of filter is it? Lowpass, highpass, bandpass, bandstop, notch, or allpass? 4 points.

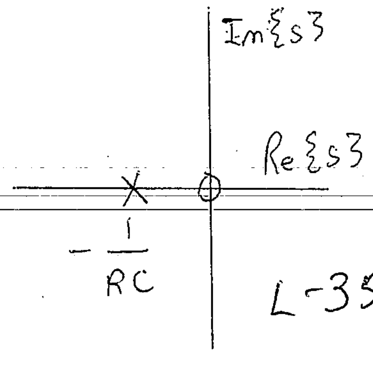
The frequency response is

$$H_{freq}(f) = H(s)\Big|_{s = j2\pi f} = \frac{j2\pi f}{j2\pi f + \frac{1}{RC}}$$

$$\boxed{\text{Highpass Filter}}$$

$$H_{freq}(0) = 0$$

$$\lim_{f \to \infty} H_{freq}(f) = \lim_{f \to \infty} \frac{j2\pi}{j2\pi + \frac{1}{RCf}} = 1$$



L-35

L-36